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On modular solutions of fractional weights for the Kaneko–Zagier equation for $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$

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Abstract For the full modular group, Kaneko gave a modular form solution of fractional weights for a holomorphic modular differential equation of second order. In this paper, we give modular form solutions of fractional weights for a modular differential equation on the Fricke group of levels 2 and 3.

Keywords Modular form · Differential equations · The Heun series · Supersingular j_N-polynomial

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1 Introduction and preliminaries

In [4], Kaneko–Koike studied various solutions for the so-called Kaneko–Zagier equation:

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0,$$

where $' = (2\pi i)^{-1}d/d\tau = qd/dq, q = e^{2\pi i\tau}$, τ a variable in the Poincaré upper-half plane, k a fixed rational number, and $E_2(\tau)$ is the (quasimodular) Eisenstein series of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^n.$$

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They gave modular forms expressed in terms of hypergeometric polynomials and quasimodular forms as its solution for weight k, where k is integer or half-integer. In particular, Kaneko studied in [3] the modular form as its solution for weight one-fifth, which is closely related to certain models in conformal field theory.

From [5,6] and [11], we know that the Kaneko–Zagier equation for $\Gamma_0^*(N)(N = 2, 3)$

$$(\sharp)_k^{(N)} \quad f''(\tau) - \frac{k+1}{6-N} E_{NA}(\tau) f'(\tau) + \frac{k(k+1)}{2(6-N)} E'_{NA}(\tau) f(\tau) = 0$$

also has modular/quasimodular solutions similar to the case for $\mathrm{SL}_2(\mathbb{Z})$, where the Fricke group of level N $(N = 2, 3)$ is defined by

$$\Gamma_0^*(N) = \Gamma_0(N) \cup \Gamma_0(N) W_N, \qquad W_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix},$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\},$$

and $E_{NA}(\tau)$, the (quasimodular) Eisenstein series of weight 2 for $\Gamma_0^*(N)$, is defined by

$$E_{NA}(\tau) = \frac{N E_2(N\tau) + E_2(\tau)}{N+1}.$$

In this paper, we give modular forms of a fractional weight as a solution of the Kaneko–Zagier equation for $\Gamma_0^*(N)(N = 2, 3)$. Hereafter, N denotes the level 2 or 3.

For any complex numbers v and s, we take $-\pi < \arg(v) \leq \pi$ and put $v^s = |v|^s e^{is\arg(v)}$. Define

$$\begin{aligned}
\phi_1^{(2)}(\tau) &= \left(\frac{\eta(\tau)}{\eta(2\tau)^2}\right)^{1/3} \frac{\eta(2\tau)\eta(3\tau)^2}{\eta(\tau)\eta(6\tau)} \\
&= 1 + \frac{2}{3}q + \frac{8}{9}q^2 - \frac{50}{81}q^3 + \frac{74}{243}q^4 + \frac{320}{729}q^5 + \frac{1232}{6561}q^6 + \frac{7012}{19683}q^7 + \cdots, \\
\phi_2^{(2)}(\tau) &= \left(\frac{\eta(\tau)}{\eta(2\tau)^2}\right)^{1/3} \frac{\eta(6\tau)^2}{\eta(3\tau)} \\
&= q^{1/3}\left(1 - \frac{1}{3}q + \frac{2}{9}q^2 + \frac{40}{81}q^3 + \frac{62}{243}q^4 - \frac{307}{729}q^5 + \frac{458}{6561}q^6 - \frac{4136}{19683}q^7 + \cdots\right), \\
\phi_1^{(3)}(\tau) &= \left(\frac{\eta(\tau)}{\eta(3\tau)^3}\right)^{1/2} \frac{\eta(2\tau)^3\eta(3\tau)^2}{\eta(\tau)^2\eta(6\tau)} \\
&= 1 + \frac{3}{2}q + \frac{3}{8}q^2 + \frac{15}{16}q^3 + \frac{3}{128}q^4 - \frac{99}{256}q^5 + \frac{1671}{1024}q^6 + \frac{1383}{2048}q^7 + \cdots, \\
\phi_2^{(3)}(\tau) &= \left(\frac{\eta(\tau)}{\eta(3\tau)^3}\right)^{1/2} \frac{\eta(6\tau)^3}{\eta(2\tau)} \\
&= q^{1/2}\left(1 - \frac{1}{2}q + \frac{3}{8}q^2 + \frac{11}{16}q^3 + \frac{35}{128}q^4 - \frac{159}{256}q^5 + \frac{359}{1024}q^6 - \frac{573}{2048}q^7 + \cdots\right),
\end{aligned}$$

where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function. Then, we find that $\phi_1^{(N)}(\tau)$ and $\phi_2^{(N)}(\tau)$ are holomorphic modular forms of weight $N/6$ for

$$\Gamma(6) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(6) \,\middle|\, a \equiv d \equiv 1,\ b \equiv 0 \pmod 6 \right\}$$

using properties of $\eta(\tau)$ [7, §1.3 Theorem 1.7 and §2.3 Corollary 2.2]. Moreover, $(\phi_1^{(N)})^{6/N}$ and $(\phi_2^{(N)})^{6/N}$ are modular forms of weight 1 with the Legendre character $\left(\frac{*}{3}\right)$ for $\Gamma_0(6)$.

Remark 1 The following can be expressed in terms of theta series:

$$\frac{\eta(2\tau)\eta(3\tau)^2}{\eta(\tau)\eta(6\tau)} = \sum_{n\in\mathbb{Z}} q^{(6n+1)^2/24}, \qquad \frac{\eta(6\tau)^2}{\eta(3\tau)} = \sum_{n\in\mathbb{Z}} q^{3(4n+1)^2/8}.$$

Note that if you find a solution $F(\phi_1^{(N)},\ \phi_2^{(N)})$ of weight k for Eq. $(\sharp)_k^{(N)}$, you can get another solution $F(\phi_2^{(N)},\ -\phi_1^{(N)}/N)$ immediately because the group $\Gamma_0^*(N)$ acts on the space of solutions as follows:

$$\begin{pmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{pmatrix}\Bigg|_{\frac{1}{3}} \begin{bmatrix} 2\sqrt{2} & -3/\sqrt{2} \\ 3\sqrt{2} & -2\sqrt{2} \end{bmatrix} = \sqrt{2}e^{-\frac{2}{3}\pi i} \begin{pmatrix} \phi_2^{(2)} \\ -\frac{1}{2}\phi_1^{(2)} \end{pmatrix},$$

$$\begin{pmatrix} \phi_1^{(3)} \\ \phi_2^{(3)} \end{pmatrix}\Bigg|_{\frac{1}{2}} \begin{bmatrix} -\sqrt{3} & -4/\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3} \end{bmatrix} = \sqrt{3}e^{-\frac{1}{4}\pi i} \begin{pmatrix} \phi_2^{(3)} \\ -\frac{1}{3}\phi_1^{(3)} \end{pmatrix},$$

where $F(X, Y)$ is a homogenous polynomial of two variables, and $|_k[\cdot]$ is a slash operator of weight k.

Finally, Heun's local series Hl is defined by

$$Hl(a, w; \alpha, \beta, \gamma, \delta; x) = \sum_{n=0}^{\infty} c_n x^n,$$

where the coefficients satisfy the recursion; $c_0 = 1,\ \ c_1 = \frac{w}{a\gamma}c_0$, and

$$c_{n+1} = \frac{(n[(n-1+\gamma)(1+a)+a\delta+\varepsilon]+w)}{(n+1)(n+\gamma)a}c_n - \frac{(n-1+\alpha)(n-1+\beta)}{(n+1)(n+\gamma)a}c_{n-1} \qquad (n \geq 1),$$

where $\gamma + \delta + \varepsilon = \alpha + \beta + 1$. This is a solution of Heun's equation, which is the canonical form of a second-order linear differential equation with four regular singularities (cf. [10]):

$$\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a}\right)\frac{dy}{dx} + \frac{\alpha\beta x - w}{x(x-1)(x-a)}y = 0. \tag{1}$$

In particular, Hl is a polynomial when α or $\beta \in -\mathbb{N}$.

2 Main result

Theorem 1 (1) *Assume $k = (4n+1)/3$ such that $n = 0, 1, 2, \ldots,\ n \not\equiv 2 \pmod 3$. Then Eq. $(\sharp)_k^{(2)}$ has a two-dimensional space of solutions in $\mathbb{C}[\phi_1^{(2)}, \phi_2^{(2)}]_{wt=k}$. Its generators are*

$$\phi_1^{(2)}(\tau)^{3k} Hl\Big(-8, \frac{k(1-3k)}{2}; -k, \frac{1-3k}{4}, \frac{3-k}{4}, \frac{1-3k}{4}; 8\frac{\phi_2^{(2)}(\tau)^3}{\phi_1^{(2)}(\tau)^3}\Big) = 1 + O(q),$$

$$\phi_2^{(2)}(\tau)^{\frac{k+1}{4}} \phi_1^{(2)}(\tau)^{\frac{11k-1}{4}} Hl\Big(-8, \frac{(k-7)(1-3k)}{16}; \frac{1-3k}{4}, \frac{1-k}{2}, \frac{k+5}{4}, \frac{1-3k}{4}; 8\frac{\phi_2^{(2)}(\tau)^3}{\phi_1^{(2)}(\tau)^3}\Big) = q^{\frac{k+1}{12}} + O(q^{\frac{k+13}{12}}).$$

(2) *Assume $k = (3n+1)/2$ such that $n = 0, 1, 2, \ldots,\ n \not\equiv 1 \pmod 2$. Then Eq. $(\sharp)_k^{(3)}$ has a two-dimensional space of solutions in $\mathbb{C}[\phi_1^{(3)}, \phi_2^{(3)}]_{wt=k}$. Its generators are*

$$\phi_1^{(3)}(\tau)^{2k} Hl\Big(9, k(2k-1); -k, \frac{1-2k}{3}, \frac{2-k}{3}, \frac{1-2k}{3}; 9\frac{\phi_2^{(3)}(\tau)^2}{\phi_1^{(3)}(\tau)^2}\Big) = 1 + O(q),$$

$$\phi_2^{(3)}(\tau)^{\frac{k+1}{3}} \phi_1^{(3)}(\tau)^{\frac{5k-1}{3}} Hl\Big(9, \frac{(k+10)(1-2k)}{9}; \frac{1-2k}{3}, \frac{2-k}{3}, \frac{k+4}{3}, \frac{1-2k}{3}; 9\frac{\phi_2^{(3)}(\tau)^3}{\phi_1^{(3)}(\tau)^3}\Big) = q^{\frac{k+1}{6}} + O(q^{\frac{k+7}{6}}).$$

Remark 2 By the conditions for weight k, the Heun local series in the above theorem become polynomials.

3 Proof

We will prove the result only for the case of level 2. The case of level 3 can be treated in a similar manner. To prove this theorem, we need the following proposition.

Proposition 1 ([10, p.18]) *If $Hl(a, w; \alpha, \beta, \gamma, \delta; x)$ is a solution of the Heun differential equation (1), then $x^{1-\gamma} Hl(a, w'; \alpha', \beta', \gamma', \delta; x)$ is also a solution of Eq. (1), where $\alpha' = \alpha + 1 - \gamma$, $\beta' = \beta + 1 - \gamma$, $\gamma' = 2 - \gamma$, $w' = (a\delta + \varepsilon)(1 - \gamma) + w$.*

Putting $f(\tau)/\phi_1^{(2)}(\tau)^{3k} = g(\tau)$ and $X = 8\phi_2^{(2)}(\tau)^3/\phi_1^{(2)}(\tau)^3$, Eq. $(\sharp)_k^{(2)}$ can be transformed into

$$g''(\tau) + \Big(-\frac{1}{4}E_{2A}(\tau) + \frac{k}{32}(X^2+28X-8)\phi_1^{(2)}(\tau)^6\Big)g'(\tau) + \frac{k(3k-1)}{256}X(X-1)(X+2)(X+8)\phi_1^{(2)}(\tau)^{12}g(\tau) = 0.$$

Using the relation of derivatives between $2\pi i\tau$ and X:

$$g'(\tau) = -\frac{1}{8}X(X-1)(X+8)\phi_1^{(2)}(\tau)^6\frac{dg}{dX},$$
$$g''(\tau) = \frac{1}{64}X^2(X-1)^2(X+8)^2\phi_1^{(2)}(\tau)^{12}\frac{d^2g}{dX^2} + \frac{1}{256}X(X-1)(X+8)\phi_1^{(2)}(\tau)^6\Big((5X^2+28X-24)\phi_1^{(2)}(\tau)^6 - 8E_{2A}(\tau)\Big)\frac{dg}{dX},$$

we have

$$\frac{d^2g}{dX^2} + \Big(\frac{1-3k}{4X} + \frac{3-k}{4(X-1)} + \frac{1-3k}{4(X+8)}\Big)\frac{dg}{dX} + \frac{k(3k-1)}{4}\cdot\frac{X+2}{X(X-1)(X+8)}g = 0.$$

Comparing this equation with Eq. (1), we can obtain Heun's solution. Using Proposition 1, we can obtain another solution, and the two solutions are polynomials, because α or β, α' or β' $\in -\mathbb{N}$. Therefore, $f = \phi_1^{(2)}(\tau)^{3k}\cdot g$ is a modular solution of $(\sharp)_k^{(2)}$.

4 The relation to supersingular j_{NA}-polynomials

Koike defined in [9] (or see [2,8,11]) supersingular j_{NA}-polynomials. He proved that for the elliptic modular invariant $j(\tau)$ over a finite field of characteristic $p > 0$ to be supersingular is equivalent to the Hauptmodul $j_{NA}(\tau)$ for $\Gamma_0^*(N)$ being supersingular. From his result with respect to a supersingular elliptic curve, we get the following definition.

Definition 1 For a prime number $p(\geq 5)$, we define the "supersingular j_{NA}-polynomials for $\Gamma_0^*(N)$" by

$$S_p^{(2A)}(X) := X^{\delta_2}(X-256)^{\varepsilon_2}\begin{cases} X^{m_2}\mathbb{F}\big(\frac{1}{8},\frac{3}{8},1,\frac{256}{X}\big) & p\equiv 1,\ 3 \pmod 8,\\ X^{m_2}\mathbb{F}\big(\frac{7}{8},\frac{5}{8},1,\frac{256}{X}\big) & p\equiv 5,\ 7 \pmod 8,\end{cases}$$
$$S_p^{(3A)}(X) := X^{\delta_3}(X-108)^{\delta_3}\begin{cases} X^{m_3}\mathbb{F}\big(\frac{1}{6},\frac{1}{3},1,\frac{108}{X}\big) & p\equiv 1 \pmod 6,\\ X^{m_3}\mathbb{F}\big(\frac{2}{3},\frac{5}{6},1,\frac{108}{X}\big) & p\equiv 5 \pmod 6,\end{cases}$$

where $m_2 = [\frac{p}{8}]$, $p-1 = 8m_2 + 2\delta_2 + 4\varepsilon_2$, $m_3 = [\frac{p}{6}]$, $p-1 = 6m_3 + 4\delta_3$, $\delta_2, \varepsilon_2, \delta_3 \in \{0,\ 1\}$, and $\mathbb{F}(\alpha,\beta,\gamma,x)$ is the hypergeometric series over a finite field of characteristic $p > 0$.

In this section, we present a certain conjecture about a reduction mod prime p of Heun polynomials.

Let

$$j_{2A}(X_2) = \frac{(8-20X_2-X_2^2)^4}{X_2(1-X_2)^3(8+X_2)^3}, \qquad j_{3A}(X_3) = \frac{(X_3+3)^6}{X_3(1-X_3)^2(9-X_3)^2}$$

be the Hauptmodul for $\Gamma_0^*(N)$ expressed in terms of $X_2 = 8\big(\phi_2^{(2)}/\phi_1^{(2)}\big)^3$ and $X_3 = 9\big(\phi_2^{(3)}/\phi_1^{(3)}\big)^2$, and further let

$$T_n^{(2)}(X_2) = Hl\Big(-8, -\frac{2n(4n+1)}{3}; -\frac{4n+1}{3}, -n, \frac{2-n}{3}, -n; X_2\Big) \text{ and}$$
$$T_n^{(3)}(X_3) = Hl\Big(9, \frac{3n(3n+1)}{2}; -\frac{3n+1}{2}, -n, \frac{1-n}{2}, -n; X_3\Big)$$

be the Heun polynomials of degree $n(>0)$.

Conjecture 1 *Let $p > 5$ be a prime. Then $T_{p-1}^{(N)}(X_N) \mod p$ is a "supersingular X_N-polynomial," i.e., it is equal to $\prod_{Y_N \in \overline{\mathbb{F}}_p}(X_N - Y_N)$, where Y_N runs through those values for which the corresponding Hauptmodul $j_{NA}(Y_N)$ is supersingular.*

5 The function like characters

From [4], we know that some solutions of the Kaneko–Zagier equation for $\mathrm{SL}_2(\mathbb{Z})$ are closely related to the character for two-dimensional conformal field theory. Precisely, we can get the character from the solution of weight k divided by $\eta(\tau)^{2k}$.

From numerical examination, for the Fricke group of levels 2 and 3, we can get something like the character from the solution $f(\tau)$ of weight k divided by $\Delta_{NA}(\tau)^{k/(12-2N)}$, where $\Delta_{2A} = \eta(\tau^8)\eta(2\tau)^8$ and $\Delta_{3A} = \eta(\tau)^6\eta(3\tau)^6$ are cusp forms for $\Gamma_0(N)$. For example, in the case for $\Gamma_0^*(2)$, we get the following:

(a) For $k = 1/3$,

$$\phi_1^{(2)}/\Delta_{2A}^{1/24} = \frac{1}{q^{\frac{1}{24}}} + q^{\frac{23}{24}} + 2q^{\frac{47}{24}} + q^{\frac{71}{24}} + 3q^{\frac{95}{24}} + 3q^{\frac{119}{24}} + 5q^{\frac{143}{24}} + 5q^{\frac{167}{24}} + 8q^{\frac{191}{24}} + \cdots;$$

the number of partitions of n in which no part appears more than twice and no two parts differ by 1.

(b) For $k = 2$,

$$\frac{(\phi_1^{(2)})^6 + 20(\phi_1^{(2)}\phi_2^{(2)})^3 - 8(\phi_2^{(2)})^6}{\Delta_{2A}^{1/4}} = \frac{1}{q^{\frac{1}{4}}} + 26q^{\frac{3}{4}} + 79q^{\frac{7}{4}} + 326q^{\frac{11}{4}} + 755q^{\frac{15}{4}} + 2106q^{\frac{19}{4}} + \cdots,$$

the McKay–Thompson series of class 8C for the Monster. (cf. [1])

For several other weights k, we observe that each coefficient of $f(\tau)/\Delta_{NA}(\tau)^{k/(12-2N)}$ is a positive integer, where $f(\tau)$ is a modular solution of weight k for $(\sharp)_k^{(N)}$. But we do not know to which the function corresponds and what are the properties of these functions.

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